

**Phys 410**  
**Fall 2015**  
**Lecture #25 Summary**  
**24 November, 2015**

We considered the most general coupled oscillator problem –  $N$  particles in three dimensions coupled to each other by means of springs or any other types of forces that produce a stable equilibrium configuration. This system has  $n$  generalized (perhaps normal) coordinates, where in general  $n \leq 3N$ . The generalized coordinates are written as  $\vec{q} = (q_1, q_2, \dots, q_n)$ . We assume that only conservative forces act between the particles, hence (as known from previous studies) the potential energy is a function only of the coordinates:  $U = U(\vec{q})$ . The kinetic energy is that of all of the particles in the system:  $T = \frac{1}{2} \sum_{\alpha=1}^N m_{\alpha} \dot{\vec{r}}_{\alpha}^2$ . The “raw” coordinates  $\vec{r}_{\alpha}$  can be written in terms of the generalized coordinates as  $\vec{r}_{\alpha} = \vec{r}_{\alpha}(q_1, q_2, \dots, q_n)$ , where it is assumed that no explicit time-dependence is required to write down this transformation. The kinetic energy can be written as  $T = \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n A_{ij} \dot{q}_i \dot{q}_j$ , where the matrix  $\bar{A}$  is defined as  $A_{ij} \equiv \sum_{\alpha=1}^N m_{\alpha} \frac{\partial \vec{r}_{\alpha}}{\partial q_i} \cdot \frac{\partial \vec{r}_{\alpha}}{\partial q_j}$ . Note that the double pendulum kinetic energy (see the Lagrangian in the last lecture) has a kinetic energy of this form, including a  $\dot{q}_1 \dot{q}_2$  term. Note that the matrix  $\bar{A}$  is a function of the generalized coordinates as well:  $\bar{A} = \bar{A}(\vec{q})$ . We now have the full Lagrangian of this generalized coupled oscillator problem  $\mathcal{L} = T(\vec{q}, \dot{\vec{q}}) - U(\vec{q})$ .

To make further progress we next considered the small oscillation motion of the system around a stable equilibrium point. This means that we will keep terms only up to second order in the variables. By a shift of the origin, we can make the stable equilibrium point appear at the point  $\vec{q} = (0, 0, \dots, 0)$ . We then did a Taylor series expansion of the potential around this point and kept terms up to second order, yielding  $U(\vec{q}) = \frac{1}{2} \sum_{i,j} K_{ij} q_i q_j$ , where the matrix elements of  $\bar{K}$  are the curvatures of the potential with respect to the generalized coordinates:  $K_{ij} \equiv \left. \frac{\partial^2 U}{\partial q_i \partial q_j} \right|_{\vec{q}=\vec{0}}$ . The kinetic energy is already quadratic in the variables, so we simply evaluate it at  $\vec{q} = \vec{0}$  to yield  $T = \frac{1}{2} \sum_{i,j} A_{ij}(0) \dot{q}_i \dot{q}_j = \frac{1}{2} \sum_{i,j} M_{ij} \dot{q}_i \dot{q}_j$ , where the mass matrix  $\bar{M}$  is the  $\bar{A}$  matrix evaluated at the equilibrium position  $\vec{q} = (0, 0, \dots, 0)$ . The Lagrangian  $\mathcal{L} = T(\dot{\vec{q}}) - U(\vec{q})$  is now a homogeneous quadratic function of the coordinates and their time-derivatives, and the matrices  $\bar{M}$  and  $\bar{K}$  are constant symmetric real matrices.

There are  $n$  Lagrange equations to set up and solve. We wrote down the equations and found that the set of  $n$  equations are summarized beautifully in a simple matrix equation:  $-\bar{K}\vec{q} = \bar{M}\ddot{\vec{q}}$ . We can solve this equation using the same method employed before, just

generalized to  $n$  coordinates. We use the standard complex *ansatz* for the solution vector:

$$\vec{q}(t) = \text{Re}[\vec{C}e^{i\omega t}], \text{ where } \vec{C} = \begin{pmatrix} C_1 \\ C_2 \\ \vdots \\ C_n \end{pmatrix}, \text{ and the } C_i \text{ are complex constants. This assumes that}$$

all of the coordinates adopt the same oscillation frequency  $\omega$ , and each oscillator will adopt its own amplitude and phase (through the choice of  $C_i = A_i e^{-i\delta_i}$ ). Putting this into the matrix equation yields  $(\bar{K} - \omega^2 \bar{M})\vec{C} = 0$ . To get a non-trivial solution for  $\vec{C}$ , we demand that  $\det(\bar{K} - \omega^2 \bar{M}) = 0$ . This yields an  $n$ -th order equation for  $\omega^2$ , with  $n$  real solutions (we know this because the matrix  $\bar{K} - \omega^2 \bar{M}$  is real and symmetric). The  $n$  normal modes follow by standard linear algebra. The most general solution is a linear combination of motion in all of the normal modes, each with distinct amplitude and phase. The motion in a given normal mode involves, in general, a coordinated motion of all the particles in the system! The Kuramoto model describes a collection of coupled oscillators, each with a unique natural frequency and coupled to all other oscillators through a nonlinear coupling. It describes the synchronization (or lack thereof) of generators making up the power grid, as well as the coordinated oscillations of [fireflies](#) in Tennessee, among other things.

We began a discussion of nonlinear mechanics using the driven damped pendulum as an example. Note that this is an open, dissipative system. It obeys a nonlinear equation of motion because the dependent variable appears as a power other than unity in the equation. In this case the  $\sin \varphi$  term is the nonlinear term. When an equation is nonlinear we are not able to rely on the principle of superposition to find the most general solution. In general even multiplying a solution by a constant does not necessarily yield another solution to the equation! (See HW 11, Problem 12.1 for a good example.) In addition, we will see that solutions to nonlinear mechanics equations are sometimes very sensitive to initial conditions.

The driven damped pendulum has an equation of motion for the generalized coordinate  $\varphi$  given by  $\ddot{\varphi} + 2\beta\dot{\varphi} + \omega_0^2 \sin \varphi = \gamma\omega_0^2 \cos \omega t$ , where  $\beta$  is the familiar damping frequency,  $\omega_0^2 = g/\ell$  is the natural oscillation frequency of the pendulum, and  $\gamma = F_0/mg$  is the dimensionless ratio of the sinusoidal forcing amplitude to the weight of the bob. For small  $\gamma \ll 1$  we expect the nonlinear term to be approximated by the linear term  $\sin \varphi \cong \varphi$  and we have the familiar driven damped harmonic oscillator problem, studied in detail earlier. However as the drive amplitude increases and  $\gamma$  approaches 1, we expect the nonlinear term to create new kinds of solutions.

We followed the [slides](#) available in the supplementary material. Many of the figures showing solutions for  $\varphi(t)$  are generated numerically using Mathematica. We assume that the driving frequency is  $\omega = 2\pi$  so that the driving period is  $T = \frac{2\pi}{\omega} = 1$  for convenience. We also assume that the driving frequency is off resonance, but not too far off,  $\omega_0 = 3\omega/2$ , and that there is moderate damping,  $\beta = \omega_0/4$ . Various initial conditions will be considered.

For small driving amplitudes  $\gamma \ll 1$  a behavior similar to the damped driven harmonic oscillator is seen,  $\phi(t) = A \cos(\omega t - \delta)$ , after some initial transient behavior. This is called the “periodic attractor,” a terminology that will be motivated later. For moderate drive amplitude ( $\gamma = 0.9$ ) one can approximate the  $\sin \varphi$  term in terms of its first two expansion terms,  $\sin \varphi \cong \varphi - \varphi^3/3!$ . This leads to third harmonic generation in the solution for  $\varphi(t)$ , a very common property of nonlinear systems, such as diodes.